L. D. Griffin, M. Lillholm (Eds.): Scale-Space 2003, LNCS 2695, pp. 86-100, June 2003.

Equivalence Results for TV Diffusion and TV Regularisation

Thomas Brox[1], Martin Welk[1], Gabriele Steidl[2], and Joachim Weickert[1]

[1] Mathematical Image Analysis Group
Faculty of Mathematics and Computer Science, Building 27
Saarland University, 66041 Saarbrücken, Germany.
{brox,welk,weickert}@mia.uni-saarland.de
http://www.mia.uni-saarland.de

[2] Faculty of Mathematics and Computer Science, D7, 27
University of Mannheim, 68131 Mannheim, Germany
steidl@math.uni-mannheim.de
http://www.kiwi.math.uni-mannheim.de

Abstract. It has been stressed that regularisation methods and diffusion processes approximate each other. In this paper we identify a situation where both processes are even identical: the space-discrete 1-D case of total variation (TV) denoising. This equivalence is proved by deriving identical analytical solutions for both processes. The temporal evolution confirms that space-discrete TV methods implement a region merging strategy with finite extinction time. Between two merging events, only extremal segments move. Their speed is inversely proportional to their size. Our results stress the distinguished nature of TV denoising. Furthermore, they enable a mutual transfer of all theoretical and algorithmic achievements between both techniques.

1 Introduction

In the present paper we are concerned with two successful signal and image restoration methods: diffusion filters and regularisation methods. Both techniques serve the same denoising purpose and both methods can be formulated in terms of partial differential equations (PDEs). This has triggered several researchers to investigate connections between both paradigms.

In order to review their results, let us start with a brief description of 1-D diffusion filtering. We consider a noisy signal as some function $f : [a, b] \to \mathbb{R}$. The basic idea behind nonlinear diffusion filtering is to obtain a family $u(x, t)$ of filtered versions of the signal $f(x)$ as the solution of a suitable diffusion process with $f(x)$ as initial condition and homogeneous Neumann boundary conditions [19]:

$$
\begin{aligned}
u_t &= (g(u_x^2)\, u_x)_x && \text{on } (a, b) \times (0, \infty), \\
u(x, 0) &= f(x) && \text{for all } x \in [a, b], \\
u_x(a, t) = u_x(b, t) &= 0 && \text{for all } t \in (0, \infty),
\end{aligned}
\tag{1}
$$

where subscripts denote partial derivatives, and larger diffusion times t correspond to more simplified signal representations.

Regularisation methods constitute an alternative to diffusion filters. Here the basic idea is to look for the minimiser u of the energy functional

$$E(u;\alpha,f) \;:=\; \int_a^b \Big((u-f)^2 \;+\; \alpha\, \Psi(u_x^2) \Big)\, dx. \tag{2}$$

The first term of this functional encourages similarity between the original signal $f(x)$ and its filtered version $u(x)$, while the second term penalises deviations from smoothness. The increasing function Ψ is called *penaliser (regulariser)*, and the nonnegative *regularisation parameter* α serves as smoothness weight: larger values correspond to a more pronounced filtering.

As is explained in detail in [23], there are strong relations between regularisation methods and diffusion filters (see also [17, 26]): A minimiser of (2) satisfies necessarily the Euler–Lagrange equation

$$\frac{u-f}{\alpha} \;=\; (\Psi'(u_x^2)\, u_x)_x,$$

with homogeneous Neumann boundary conditions. This equation may be regarded as a fully implicit time discretisation of the diffusion equation (1) with diffusivity $g(u_x^2) = \Psi'(u_x^2)$, initial value $f(x)$, and stopping time $t = \alpha$. Thus, one would expect that the minimiser of (2) *approximates* the diffusion filter (1), but is not identical to it. In [21, 23] this relation has been used to establish scale-space properties for regularisation methods that resemble results for diffusion filters in [28].

While the before mentioned situation is an *approximation* only, one may be interested in results where one can establish *equivalence* between a diffusion filter and a regularisation method.

Nielsen et al. [18] have shown that the solution of the linear diffusion filter [14, 29]

$$\begin{aligned} u_t &= u_{xx} \\ u(x,0) &= f(x) \end{aligned}$$

at time $t = \alpha$ may be regarded as the exact minimiser of an energy functional with an infinite number of penalising terms of arbitrarily high order:

$$E(u;\alpha,f) = \int_{\mathbb{R}} \left[(u-f)^2 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \left(\frac{d^k u}{dx^k} \right)^2 \right] dx.$$

An equivalent result has also been obtained earlier by Yuille and Grzywacz in the context of visual motion perception [30, 31].

Another linear PDE-based filter is given by the pseudodifferential equation [10, 9]

$$u_t = -\sqrt{-\frac{\partial^2}{\partial x^2}}\, u$$
$$u(x,0) = f(x).$$

Duits et al. [9] have shown that this so-called *Poisson scale-space* may be regarded as the exact minimiser of

$$E(u;\alpha,f) = \int_{\mathbb{R}} \left[(f-u)^2 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \left(\left(-\frac{d^2}{dx^2} \right)^{k/4} u \right)^2 \right] dx.$$

This discussion illustrates that only in the *linear* case, people have been able to derive energy functionals that are equivalent to the evolution equation. Unfortunately, these functionals are relatively complicated, since they involve an infinite number of regularising terms. This gives rise to the question if it is possible to derive also equivalence results in the *nonlinear* setting. Moreover, it would be nice if these energy functionals had simple regularisers that do not involve a large number of high-order derivatives.

We will address these problems in the present paper. To keep things as simple as possible, we will focus on the spatially discrete 1-D case. Surprisingly, it turns out that there exists a nonlinear framework in which there is an equivalence between diffusion filtering and regularisation that has a significantly simpler structure than equivalences in the linear case. This framework is given by so-called total variation (TV) denoising methods [22, 2]. By deriving analytical solutions that are identical for TV diffusion and TV regularisation, we prove the equivalence of both paradigms.

Our paper is organised as follows. Section 2 gives an introduction to the continuous formulations of TV diffusion and TV regularisation. In Section 3 we shall derive the analytical solution for space-discrete TV diffusion, and in Section 4 we use the same proof structure to find an identical analytical solution for discrete TV regularisation. The paper will be concluded with a summary in Section 5.

Related Work. Space-discrete nonlinear TV diffusion creates a dynamical system with a discontinuous right hand side. Systems of this type – but with different force functions – have been proposed by Pollak et al. [20] for solving segmentation problems. Analytical results for some convex regularisation problems applied to specific test signals have been presented by Li [15]. Strong [25] derived analytical results in the case of continuous TV regularisation methods with step functions as initialisations. Equivalent results have been obtained by Mammen and van de Geer [16] for the taut-string algorithm in statistics; see also [13]. Our results are in accordance with these findings, but our proof shows that they can be derived in a different way: The structure of our proof is in complete analogy with the proof for the TV diffusion case. The results in the present paper can also be extended to investigate situations in which TV denoising methods are equivalent to wavelet shrinkage techniques. This is investigated in [24].

2 Continuous TV Diffusion and TV Regularisation

2.1 TV Diffusion

One-dimensional TV diffusion is a nonlinear diffusion filter that uses the unbounded diffusivity $g(u_x^2) = 1/|u_x|$. Hence it is based on the equation

$$u_t = \left(\frac{u_x}{|u_x|}\right)_x$$

This equation has been considered by Andreu et al. [2] under the name *total variation flow*. It requires no additional parameters (besides t), it is well-posed [2, 4, 11], it preserves the shape of some objects [4], and it leads to constant signals in finite time [3]. A numerical algorithm based on level sets has been proposed in [8].

2.2 TV Regularisation

TV regularisation has been proposed in its unconstrained form by Rudin, Osher and Fatemi [22], and in its constrained form by Acar and Vogel [1]. It uses the penaliser $\Psi(u_x^2) = 2|u_x|$. Hence, the constrained form minimises

$$E(u;\alpha,f) \;:=\; \int_a^b \Big((u-f)^2 \;+\; 2\alpha|u_x|\Big)\,dx$$

This regularisation strategy is well-known for its good denoising capabilities and its tendency to create blocky, segmentation-like results. Well-posedness results have been established in [5]. A number of numerical schemes have been proposed including primal-dual methods [6], nonlinear Jacobi algorithms [7], and multigrid strategies [27].

3 Analytical Solution for Space-Discrete TV Diffusion

Let us now consider a space-discrete formulation of TV diffusion. We assume that the spatial grid size is 1 and that $f = (f_0, \ldots, f_{N-1})$ denotes a discrete version of $f(x)$ with N pixels. This leads to the following dynamical system:

$$\left.\begin{aligned} \dot{u}_0 &= \operatorname{sgn}(u_1 - u_0), \\ \dot{u}_i &= \operatorname{sgn}(u_{i+1} - u_i) - \operatorname{sgn}(u_i - u_{i-1}) \qquad (i = 1, \ldots, N-2), \\ \dot{u}_{N-1} &= -\operatorname{sgn}(u_{N-1} - u_{N-2}), \\ u(0) &= f. \end{aligned}\right\} \tag{3}$$

In the following, we further set $u_{-1} := u_0$ and $u_N := u_{N-1}$, which may be regarded as a discretisation of the homogeneous Neumann boundary conditions. Since the right-hand side of this system is discontinuous, we need a more detailed specification of when a system of functions is said to satisfy these differential equations (cf. also [12]). A vector-valued function u is said to fulfil the system (3) over the time interval $[0, T]$ if the following holds true:

(I) u is an absolutely continuous vector-valued function which satisfies (3) almost everywhere, where sgn is defined by $\operatorname{sgn} w := 1$ if $w > 0$, $\operatorname{sgn} w := -1$ if $w < 0$, and may take any value in $[-1, 1]$ if $w = 0$.
(II) If $\dot{u}_i(t)$ and $\dot{u}_{i+1}(t)$ exist for the same t, and $u_{i+1}(t) = u_i(t)$ holds, then the expression $\operatorname{sgn}(u_{i+1}(t) - u_i(t))$ occurring in both the right-hand sides for $\dot{u}_i(t)$ and $\dot{u}_{i+1}(t)$ must take the same value in both equations.

Under these conditions we obtain the following result:

Proposition 1. (Properties of Space-Discrete TV Diffusion)
The system (3) has a unique solution $u(t)$ in the sense of (I) and (II). This solution has the following properties:

(i) (Finite Extinction Time)
There exists a finite time $T \geq 0$ such that for all $t \geq T$ the signal becomes constant:

$$u_i(t) = \frac{1}{N} \sum_{k=0}^{N-1} f_k \qquad \text{for all } i = 0, \ldots, N-1.$$

(ii) (Finite Number of Merging Events)
There exists a finite sequence $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ such that the interval $[0, T)$ splits into sub-intervals $[t_j, t_{j+1})$ with the property that for all $i = 0, \ldots, N-2$ either $u_i(t) = u_{i+1}(t)$ or $u_i(t) \neq u_{i+1}(t)$ throughout $[t_j, t_{j+1})$. The absolute difference between neighbouring pixels does not become larger for increasing $t \in [t_j, t_{j+1})$.

(iii) (Analytical Solution)
In each of the sub-intervals $[t_j, t_{j+1})$ constant regions of $u(t)$ evolve linearly: For a fixed index i let us consider a constant region given by

$$u_{i-l+1} = \ldots = u_i = u_{i+1} = \ldots = u_{i+r} \qquad (l \geq 1, r \geq 0) \tag{4}$$

and

$$u_{i-l} \neq u_{i-l+1} \ \text{ if } i - l \geq 0, \quad u_{i+r} \neq u_{i+r+1} \ \text{ if } i + r \leq N - 1$$

for all $t \in [t_j, t_{j+1})$. We call (4) a region of size $m_{i,t_j} = l + r$. For $t \in [t_j, t_{j+1})$ let $\triangle t = t - t_j$. Then $u_i(t)$ is given by

$$u_i(t) = u_i(t_j) + \mu_{i,t_j} \frac{2\triangle t}{m_{i,t_j}},$$

where μ_{i,t_j} reflects the relation between the region containing u_i and its neighbouring regions. It is given as follows:
For inner regions (i.e. $i-l \geq 0$ and $i+r \leq N-1$) we have

$$\mu_{i,t_j} = \begin{cases} 0 \text{ if } (u_{i-l}, u_i, u_{i+r+1}) \textit{ is strictly monotonous}, \\ 1 \text{ if } u_i \textit{ is minimal in } (u_{i-l}, u_i, u_{i+r+1}), \\ -1 \text{ if } u_i \textit{ is maximal in } (u_{i-l}, u_i, u_{i+r+1}) \end{cases} \tag{5}$$

and in the boundary case ($i-l+1=0$ or $i+r=N-1$), the evolution is half as fast:

$$\mu_{i,t_j} = \begin{cases} 0 \text{ if } m = N, \\ \frac{1}{2} \text{ if } u_i \textit{ is minimal in } (u_{i-l}, u_i, u_{i+r+1}), \\ -\frac{1}{2} \text{ if } u_i \textit{ is maximal in } (u_{i-l}, u_i, u_{i+r+1}). \end{cases} \tag{6}$$

Proof.
Let u be a solution of (3). We show that u is uniquely determined and satisfies the rules (i)–(iii). Our proof proceeds in four steps.

1. If $\dot{u}(t)$ exists at a fixed time t and $u_i(t)$ lies at this time in some region

$$u_{i-l+1}(t) = \ldots = u_i(t) = \ldots = u_{i+r}(t) \qquad (l \geq 1, r \geq 0),$$

$$u_{i-l}(t) \neq u_{i-l+1}(t) \;\text{ if } i-l \geq 0, \quad u_{i+r}(t) \neq u_{i+r+1}(t) \;\text{ if } i+r \leq N-1$$

of size $m_{i,t}$, then it follows by (3) and (II) in the non-boundary case $i-l \geq 0$ and $i+r \leq N-1$ that

$$u_i(t) \;=\; \frac{1}{m_{i,t}} \sum_{k=-l+1}^{r} u_{i+k}(t),$$

and therefore

$$\begin{aligned} \dot{u}_i(t) &= \frac{1}{m_{i,t}} \sum_{k=-l+1}^{r} \dot{u}_{i+k}(t) \\ &= \frac{1}{m_{i,t}} \left(\operatorname{sgn}\left(u_{i+r+1}(t) - u_i(t)\right) \,-\, \operatorname{sgn}\left(u_i(t) - u_{i-l}(t)\right)\right) \\ &= \mu_{i,t} \, \frac{2}{m_{i,t}} \,, \end{aligned} \tag{7}$$

where $\mu_{i,t}$ describes the relation between the region containing u_i and its neighbours at time t as in (5). In the boundary case $i-l+1=0$ or $i+r=N-1$ we follow the same lines and obtain (7) with $\mu_{i,t}$ defined by (6).

2. Let $\dot{u}(t)$ exist in some small interval (τ_0, τ_1) and assume that $u_i(t) \neq u_{i+1}(t)$ for some $i \in \{0, \dots, N-2\}$ and all $t \in (\tau_0, \tau_1)$. By continuity of u we may assume that $u_i(t) < u_{i+1}(t)$ throughout (τ_0, τ_1). The opposite case $u_i(t) > u_{i+1}(t)$ can be handled in the same way. Then we obtain by (7) and definition of $\mu_{i,t}$ for all $t \in (\tau_0, \tau_1)$ that

$$\dot{u}_i(t) \geq 0 \quad \text{if} \quad i - l \geq 0, \tag{8}$$

$$\dot{u}_i(t) > 0 \quad \text{if} \quad i - l + 1 = 0, \tag{9}$$

$$\dot{u}_{i+1}(t) \leq 0 \quad \text{if} \quad i + r \leq N - 2, \tag{10}$$

$$\dot{u}_{i+1}(t) < 0 \quad \text{if} \quad i + r = N - 1. \tag{11}$$

Set $w(t) := u_{i+1}(t) - u_i(t)$. Then the mean value theorem yields

$$w(\tau_1) - w(\tau_0) = (\tau_1 - \tau_0)\, \dot{w}(t^*)$$

for some $t^* \in (\tau_0, \tau_1)$ and we get by (8)–(11) that

$$w(\tau_1) - w(\tau_0) \leq 0$$

with strict inequality in the boundary case. Consequently, the difference between pixels cannot become larger in the considered interval. In particular, by continuity of u, pixels cannot be split. Once merged they stay merged.

3. Now we start at time $t_0 = 0$. Let t_1 be the largest time such that $\dot{u}(t)$ exists and no merging of regions appears in $(0, t_1)$. Then, for all $i \in \{0, \dots, N-1\}$, a function u_i is in the same region with the same relations to its neighbouring regions throughout $[0, t_1)$. Thus, we conclude by (7) that

$$\dot{u}_i(t) = \mu_{i,0}\, \frac{2}{m_{i,0}} \qquad (t \in (0, t_1))$$

and consequently

$$\begin{aligned} u_i(t) &= \mu_{i,0}\, \frac{2t}{m_{i,0}} + C_{i,0} \\ &= f_i \,+\, \mu_{i,0}\, \frac{2t}{m_{i,0}} \qquad (t \in [0, t_1]), \end{aligned}$$

where the last equality follows by continuity of u_i if t approaches 0.

4. We are now in the position to analyse the entire chain of merging events successively.
Next we consider the largest interval (t_1, t_2) without merging events in the same way, where we take the initial setting $u(t_1)$ instead of f into account. Then we obtain

$$u_i(t) = \mu_{i,t_1}\, \frac{2t}{m_{i,t_1}} + C_{i,t_1},$$

where $u_i(t_1) = \mu_{i,t_1}\, \frac{2t_1}{m_{i,t_1}} + C_{i,t_1}$ by continuity of u_i. Consequently

$$u_i(t) = u_i(t_1) + \mu_{i,t_1}\, \frac{2(t - t_1)}{m_{i,t_1}}\,.$$

Now we can continue in the same way by considering $[t_2, t_3)$ and so on. Since we have only a finite number N of pixels and some of these pixels merge at the points t_j the process stops after a finite number of n steps with output

$$u_i(t_n) = \frac{1}{N} \sum_{k=0}^{N-1} f_k$$

for all $i = 0, \ldots, N-1$.

Conversely, it is easy to check that a function u with (i)–(iii) is a solution of the system (3). This completes the proof of the proposition. □

4 Analytical Solution for Discrete TV Regularisation

Next we will prove that discrete TV regularisation satisfies the same rules as space-discrete TV diffusion. For given initial data $f = (f_0, \ldots, f_{N-1})$ discrete TV regularisation consists in constructing the minimiser

$$u(\alpha) = \min_u E(u; \alpha, f) \tag{12}$$

of the functional

$$E(u; \alpha, f) = \sum_{i=0}^{N-1} \left((u_i - f_i)^2 + 2\alpha \, |u_{i+1} - u_i| \right), \tag{13}$$

where we suppose again Neumann boundary conditions $u_{-1} = u_0$ and $u_N = u_{N-1}$.

For a fixed regularisation parameter $\alpha \geq 0$, the minimiser of (13) is uniquely determined since $E(u; \alpha, f)$ is strictly convex in $u_0, \ldots, u_{N-1}$. Furthermore, $E(u, \alpha; f)$ is a continuous function in $u_0, \ldots, u_{N-1}, \alpha$. Consequently, $u(\alpha)$ is a (componentwise) continuous function in α.

The following proposition implies together with Proposition 1 the equivalence of space-discrete TV diffusion and discrete TV regularisation, if the diffusion time t is identical to the regularisation parameter α.

Proposition 2. (Properties of Discrete TV Regularisation)
The function $u(\alpha)$ in (12) is uniquely determined by the following rules:

(i) (Finite Extinction Parameter)
There exists a finite $A \geq 0$ such that for all $\alpha \geq A$ the signal becomes constant:

$$u_i(\alpha) = \frac{1}{N} \sum_{k=0}^{N-1} f_k \qquad \text{for all } i = 0, \ldots, N-1.$$

(ii) (Finite Number of Merging Events)
There exists a finite sequence $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n = A$ *such that the interval* $[0, A)$ *splits into sub-intervals* $[a_j, a_{j+1})$ *with the property that for all* $i = 0, \ldots, N-2$ *either* $u_i(\alpha) = u_{i+1}(\alpha)$ *or* $u_i(\alpha) \neq u_{i+1}(\alpha)$ *throughout* $[a_j, a_{j+1})$*. The absolute difference between neighbouring pixels does not become larger for increasing* $\alpha \in [a_j, a_{j+1})$*.*

(iii) (Analytical Solution)
In each of the sub-intervals $[a_j, a_{j+1})$ *constant regions of* $u(\alpha)$ *evolve linearly: For a fixed index* i *let us consider a constant region given by*

$$u_{i-l+1} = \ldots = u_i = u_{i+1} = \ldots = u_{i+r} \qquad (l \geq 1, r \geq 0) \tag{14}$$

and

$$u_{i-l} \neq u_{i-l+1} \text{ if } i-l \geq 0, \quad u_{i+r} \neq u_{i+r+1} \text{ if } i+r \leq N-2 \tag{15}$$

for all $\alpha \in [a_j, a_{j+1})$*. We call (14) a region of size* $m_{i,a_j} = l + r$*. For* $\alpha \in [a_j, a_{j+1})$ *let* $\triangle\alpha = \alpha - a_j$*.*
Then $u_i(\alpha)$ *is given by*

$$u_i(\alpha) = u_i(a_j) + \mu_{i,a_j} \frac{2\triangle\alpha}{m_{i,a_j}},$$

where μ_{i,a_j} *reflects the relation between the region containing* u_i *and its neighbouring regions. It is given as follows:*
For inner regions (i.e. $i - l \geq 0$ *and* $i + r \leq N - 2$*) we have*

$$\mu_{i,a_j} = \begin{cases} 0 \text{ if } (u_{i-l}, u_i, u_{i+r+1}) \textit{ is strictly monotonous}, \\ 1 \text{ if } u_i \textit{ is minimal in } (u_{i-l}, u_i, u_{i+r+1}), \\ -1 \text{ if } u_i \textit{ is maximal in } (u_{i-l}, u_i, u_{i+r+1}) \end{cases} \tag{16}$$

and in the boundary case ($i - l + 1 = 0$ *or* $i + r = N - 1$*), the evolution is half as fast:*

$$\mu_{i,a_j} = \begin{cases} 0 \text{ if } m = N, \\ \frac{1}{2} \text{ if } u_i \textit{ is minimal in } (u_{i-l}, u_i, u_{i+r+1}), \\ -\frac{1}{2} \text{ if } u_i \textit{ is maximal in } (u_{i-l}, u_i, u_{i+r+1}). \end{cases} \tag{17}$$

Proof:
Again our proof proceeds in four steps. It has a similar structure as the proof of Proposition 1.

1. Let us first verify the solution $u(\alpha)$ of (12) for an arbitrary but fixed $\alpha > 0$. If $u_i(\alpha)$ is contained in some region of size $m_{i,\alpha}$ with (14), (15), then, in case

$i-l \geq 0$ and $i+r \leq N-2$, we have that $u(\alpha)$ can be obtained as minimiser of

$$\begin{aligned} &E(u_0,\dots,u_{i-l},u_i,u_{i+r+1},\dots,u_{N-1};\alpha,f) \\ &= \sum_{k=-l+1}^{r} (u_i - f_{i+k})^2 + 2\alpha\left(|u_i - u_{i-l}| + |u_{i+r+1} - u_i|\right) \\ &\quad + F(u_0,\dots,u_{i-l},u_{i+r+1},\dots,u_{N-1}) \end{aligned}$$

with some function F independent of u_i. By (14), (15) the partial derivative of E with respect to u_i exists and is given by

$$\frac{\partial E}{\partial u_i} = 2 \sum_{k=-l+1}^{r} (u_i - f_{i+k}) \,-\, 4\alpha\mu_{i,\alpha}.$$

Here $\mu_{i,\alpha}$ describes the relation between the region containing u_i and its neighbours for the regularisation parameter α as in (16). Setting the partial derivative to zero, we obtain that

$$u_i(\alpha) = \frac{1}{m_{i,\alpha}} \sum_{k=-l+1}^{r} f_{i+k} \,+\, \mu_{i,\alpha}\frac{2\alpha}{m_{i,\alpha}}\,. \tag{18}$$

In the boundary case $i-l+1=0$ or $i+r=N-1$ we follow the same lines and obtain (18) with $\mu_{i,\alpha}$ defined by (17).

2. Next we show that initially merged pixels will not be split for any α in a small interval $[0,a_1]$.
For $\alpha = 0$ we have that $u(0) = f$. Let $f_i = u_i(0)$ be contained in some region of the form

$$f_{i-l_0+1} = \ldots = f_i = f_{i+1} = \ldots = f_{i+r_0} \qquad (l_0, r_0 \geq 1)$$

and

$$f_{i-l_0} \neq f_{i-l_0+1} \;\text{ if } i-l_0 \geq 0, \quad f_{i+r_0} \neq f_{i+r_0+1} \;\text{ if } i+r_0 \leq N-2.$$

By continuity of $u(\alpha)$ we can choose $\alpha_1 > 0$ so that $u_i(\alpha) \neq u_{i-l_0}(\alpha)$ and $u_{i+1}(\alpha) \neq u_{i+r_0}(\alpha)$ throughout $[0,\alpha_1)$. Assume that there exists $\alpha \in (0,\alpha_1)$ so that $u_i(\alpha) \neq u_{i+1}(\alpha)$, where we may assume that

$$u_i(\alpha) < u_{i+1}(\alpha). \tag{19}$$

The opposite case $u_i(\alpha) > u_{i+1}(\alpha)$ can be handled in the same way. Note that at time α more pixels than u_i and u_{i+1} may be separated. However, we have by (18) with some $1 \leq l \leq l_0$ and some $1 \leq r \leq r_0$ that

$$\begin{aligned} u_i(\alpha) &= \frac{1}{l} \sum_{k=-l+1}^{0} f_{i+k} \,+\, \mu_{i,\alpha}\frac{2\alpha}{l} \;=\; f_i \,+\, \mu_{i,\alpha}\frac{2\alpha}{l}\,, \\ u_{i+1}(\alpha) &= \frac{1}{r} \sum_{k=1}^{r} f_{i+k} \,+\, \mu_{i+1,\alpha}\frac{2\alpha}{r} \;=\; f_i \,+\, \mu_{i+1,\alpha}\frac{2\alpha}{r}\,, \end{aligned}$$

where we see by (19) and (16), (17) that $\mu_{i,\alpha} \geq 0$ and $\mu_{i+1,\alpha} \leq 0$. Thus, $u_i(\alpha) \geq u_{i+1}(\alpha)$ which contradicts (19). Consequently $u_i(\alpha) = u_{i+1}(\alpha)$ throughout $[0, \alpha_1)$, i.e., the pixels of our initial region stay merged.
Let $a_1 > 0$ denote the largest number such that no merging of regions appears in $[0, a_1)$. Then we have for all $i = 0, \ldots, N-1$ and all $\alpha \in [0, a_1)$ that $\mu_{i,\alpha} = \mu_{i,0}$ and regarding that $u(\alpha)$ is continuous that

$$u_i(\alpha) = f_i \; + \; \mu_{i,0} \, \frac{2\alpha}{m_{i,0}} \qquad (\alpha \in [0, a_1]). \tag{20}$$

3. Now we show that the absolute difference between neighbouring regions cannot become larger with increasing $\alpha \in [0, a_1)$.
Without loss of generality let for some fixed index i

$$u_{i-l+1} = \ldots = u_i < u_{i+1} = \ldots = u_{i+r} \qquad (l, r \geq 1)$$

and

$$u_{i-l} \neq u_{i-l+1} \text{ if } i-l \geq 0, \quad u_{i+r} \neq u_{i+r+1} \text{ if } i+r \leq N-2.$$

We consider the non-boundary case $i - l \geq 0$ and $i + r \leq N-2$ first. By (20) we obtain for $\alpha + \delta \in [0, a_1)$, $\delta > 0$ that

$$\begin{aligned} d_i(\alpha) = u_{i+1}(\alpha) - u_i(\alpha) \;&=\; f_{i+1} - f_i + 2\alpha \left(\frac{\mu_{i+1,0}}{r} - \frac{\mu_{i,0}}{l} \right), \\ d_i(\alpha+\delta) = u_{i+1}(\alpha+\delta) - u_i(\alpha+\delta) \;&=\; f_{i+1} - f_i + 2(\alpha+\delta) \left(\frac{\mu_{i+1,0}}{r} - \frac{\mu_{i,0}}{l} \right) \end{aligned}$$

and consequently

$$d_i(\alpha+\delta) - d_i(\alpha) = 2\delta \left(\frac{\mu_{i+1,0}}{r} - \frac{\mu_{i,0}}{l} \right).$$

By (16) it follows that

$$\frac{\mu_{i+1,0}}{r} - \frac{\mu_{i,0}}{l} = \begin{cases} 0 & \text{if } u_{i-l} < u_i < u_{i+1} < u_{i+r+1}, \\ -\frac{1}{r} & \text{if } u_{i-l} < u_i \text{ and } u_{i+1} > u_{i+r+1}, \\ -\frac{1}{l} & \text{if } u_{i-l} > u_i \text{ and } u_{i+1} < u_{i+r+1}, \\ -\frac{1}{r} - \frac{1}{l} & \text{if } u_{i-l} > u_i \text{ and } u_{i+1} > u_{i+r+1} \end{cases}$$

which yields the desired property $d_i(\alpha) \geq d_i(\alpha + \delta)$.
In case of boundary regions we follow the same lines but replace (16) by (17). Then we see that the absolute difference between neighbouring regions becomes smaller with increasing $\alpha \in [0, a_1)$.

4. We are now in the position to analyse the entire chain of merging events successively.
For $\alpha > a_1$ and $\triangle\alpha = \alpha - a_1$, we consider

$$\tilde{u}_i(\triangle\alpha) = \min_u E(u; \triangle\alpha, u(a_1)).$$

We can repeat the same considerations as in Part 2 of the proof but with initial setting $u(a_1)$ instead of f. It follows that there exists a_2 such that for all $i = 0, \dots, N-2$ either $\tilde{u}_i(\triangle\alpha) = \tilde{u}_{i+1}(\triangle\alpha)$ or $\tilde{u}_i(\triangle\alpha) \neq \tilde{u}_{i+1}(\triangle\alpha)$ throughout $[a_1, a_2)$, where the absolute difference between neighbouring pixels does not become larger for increasing $\triangle\alpha$. Further, we obtain by (20) and (18) that

$$\begin{aligned}\tilde{u}_i(\triangle\alpha) &= u_i(a_1) + \mu_{i,a_1}\frac{2\triangle\alpha}{m_{i,a_1}} \\ &= \frac{1}{m_{i,a_1}}\sum_{j\in R_{i,a_1}} f_j + \mu_{i,a_1}\frac{2a_1}{m_{i,a_1}} + \mu_{i,a_1}\frac{2\triangle\alpha}{m_{i,a_1}},\end{aligned}$$

where $R_{i,\alpha} = \{j : u_j(\alpha) \text{ is in the region of } u_i(\alpha)\}$ while m_{i,a_1} denotes the size of the region containing $u_i(a_1)$ and μ_{i,a_1} reflects the relation between the region containing $u_i(a_1)$ and its neighbouring regions. Since the relations between regions do not change for $\triangle\alpha \in [0, a_2 - a_1)$ we can rewrite $\tilde{u}_i(\triangle\alpha)$ as

$$\begin{aligned}\tilde{u}_i(\triangle\alpha) &= \frac{1}{m_{i,a_1+\triangle\alpha}}\sum_{j\in R_{i,a_1+\triangle\alpha}} f_j + \mu_{i,a_1+\triangle\alpha}\frac{2(a_1+\triangle\alpha)}{m_{i,a_1+\triangle\alpha}} \\ &= \frac{1}{m_{i,\alpha}}\sum_{j\in R_{i,\alpha}} f_j + \mu_{i,\alpha}\frac{2\alpha}{m_{i,\alpha}}.\end{aligned}$$

On the other hand, we have by (18) that

$$u_i(\alpha) = \frac{1}{m_{i,\alpha}}\sum_{j\in R_{i,\alpha}} f_j + \mu_{i,\alpha}\frac{2\alpha}{m_{i,\alpha}}\,.$$

Thus, $u_i(\alpha) = \tilde{u}_i(\triangle\alpha)$.
Now we can continue in the same way by considering $[a_2, a_3)$ and so on. Since we have only a finite number N of pixels and some of these pixels merge at the points a_j the process stops after a finite number of n steps with output $u(a_n)$ which by (18) reads as

$$u_i(a_n) = \frac{1}{N}\sum_{k=0}^{N-1} f_k$$

for all $i = 0, \dots, N-1$. This completes the proof. □

5 Conclusions

In this article we have seen that in the 1-D case, space discrete TV diffusion and discrete TV regularisation are identical, if we identify the diffusion time with the regularisation parameter. Given the relatively complicated relations between the linear Gaussian and Poisson scale-spaces and their corresponding regularisation methods, these results may seem to be of surprising simplicity. However, they are natural consequences from the simple structure of the scale-space evolutions of both TV processes: The evolution can be regarded as a sequence of region merging events. Between two mergings, only extremal segments are allowed to move. Their velocity is proportional to the inverse of the pixel number, and it can be guaranteed that all segments merge within a finite extinction time. These properties are more transparent than those of most other discrete scale-space evolutions and put space-discrete TV denoising in an extraordinary position: It is not only a nonlinear scale-space that preserves discontinuities, it also does not require any additional parameters, it implements a multiscale segmentation, and – last but not least – it is equivalent to its corresponding regularisation method. We conjecture that the latter property also holds for the continuous TV diffusion and regularisation process. Moreover, we are investigating if it can be extended to the higher dimensional situation. If this is the case, the door will be opened for a direct transfer between the results of two previously separated worlds: a parabolic scale-space world and an elliptic regularisation world.

Acknowledgements

The research in this paper is partly funded by the projects WE 2602/1-1 and WE 2602/2-1 of the *Deutsche Forschungsgemeinschaft (DFG)*. This is gratefully acknowledged.

References

1. R. Acar and C. R. Vogel. Analysis of bounded variation penalty methods for ill–posed problems. *Inverse Problems*, 10:1217–1229, 1994.
2. F. Andreu, C. Ballester, V. Caselles, and J. M. Mazón. Minimizing total variation flow. *Differential and Integral Equations*, 14(3):321–360, Mar. 2001.
3. F. Andreu, V. Caselles, J. I. Diaz, and J. M. Mazón. Qualitative properties of the total variation flow. *Journal of Functional Analysis*, 188(2):516–547, Feb. 2002.
4. G. Bellettini, V. Caselles, and M. Novaga. The total variation flow in R^N. *Journal of Differential Equations*, 184(2):475–525, 2002.
5. A. Chambolle and P.-L. Lions. Image recovery via total variation minimization and related problems. *Numerische Mathematik*, 76:167–188, 1997.
6. T. F. Chan, G. H. Golub, and P. Mulet. A nonlinear primal–dual method for total-variation based image restoration. In M.-O. Berger, R. Deriche, I. Herlin, J. Jaffré, and J.-M. Morel, editors, *ICAOS '96: Images, Wavelets and PDEs*, volume 219 of *Lecture Notes in Control and Information Sciences*, pages 241–252. Springer, London, 1996.
7. T. F. Chan, S. Osher, and J. Shen. The digital TV filter and nonlinear denoising. *IEEE Transactions on Image Processing*, 10(2):231–241, Feb. 2001.
8. F. Dibos and G. Koepfler. Global total variation minimization. *SIAM Journal on Numerical Analysis*, 37(2):646–664, 2000.

9. R. Duits, L. Florack, J. de Graaf, and B. ter Haar Romeny. On the axioms of scale space theory. Technical Report 2002-01, Dept. of Biomedical Engineering, TU Eindhoven, The Netherlands, Jan. 2002.
10. M. Felsberg and G. Sommer. Scale-adaptive filtering derived from the Laplace equation. In B. Radig and S. Florczyk, editors, *Pattern Recognition*, volume 2032 of *Lecture Notes in Computer Science*, pages 95–106. Springer, Berlin, 2001.
11. X. Feng and A. Prohl. Analysis of total variation flow and its finite element approximations. Technical Report 1864, Institute of Mathematics and its Applications, University of Minnesota, Minneapolis, MN, July 2002. Submitted to Communications on Pure and Applied Mathematics.
12. A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides.* Kluwer, Dordrecht, 1988.
13. W. Hinterberger, M. Hintermüller, K. Kunisch, M. von Oehsen, and O. Scherzer. Tube methods for BV regularization. Technical Report 6, Department of Computer Science, University of Innsbruck, Austria, Dec. 2002.
14. T. Iijima. Basic theory on normalization of pattern (in case of typical one-dimensional pattern). *Bulletin of the Electrotechnical Laboratory*, 26:368–388, 1962. In Japanese.
15. S. Z. Li. Close-form solution and parameter selection for convex minimization-based edge-preserving smoothing. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 20(9):916–932, Sept. 1998.
16. E. Mammen and S. van de Geer. Locally adaptive regression splines. *Annals of Statistics*, 25(1):387–413, 1997.
17. J.-M. Morel and S. Solimini. *Variational Methods in Image Segmentation.* Birkhäuser, Basel, 1994.
18. M. Nielsen, L. Florack, and R. Deriche. Regularization, scale-space and edge detection filters. *Journal of Mathematical Imaging and Vision*, 7:291–307, 1997.
19. P. Perona and J. Malik. Scale space and edge detection using anisotropic diffusion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12:629–639, 1990.
20. I. Pollak, A. S. Willsky, and H. Krim. Image segmentation and edge enhancement with stabilized inverse diffusion equations. *IEEE Transactions on Image Processing*, 9(2):256–266, Feb. 2000.
21. E. Radmoser, O. Scherzer, and J. Weickert. Scale-space properties of nonstationary iterative regularization methods. *Journal of Visual Communication and Image Representation*, 11(2):96–114, June 2000.
22. L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.
23. O. Scherzer and J. Weickert. Relations between regularization and diffusion filtering. *Journal of Mathematical Imaging and Vision*, 12(1):43–63, Feb. 2000.
24. G. Steidl, J. Weickert, T. Brox, P. Mrázek, and M. Welk. On the equivalence of soft wavelet shrinkage, total variation diffusion, total variation regularization, and SIDEs. Technical Report 26, Series SPP-1114, Department of Mathematics, University of Bremen, Germany, Feb. 2003.
25. D. M. Strong. *Adaptive Total Variation Minimizing Image Restoration.* PhD thesis, Department of Mathematics, University of California, Los Angeles, CA, 1997.
26. D. M. Strong and T. F. Chan. Relation of regularization parameter and scale in total variation based image denoising. Technical Report CAM-96-7, Department of Mathematics, University of California at Los Angeles, CA, U.S.A., 1996.
27. C. R. Vogel. *Computational Methods for Inverse Problems.* SIAM, Philadelphia, 2002.

28. J. Weickert. *Anisotropic Diffusion in Image Processing*. Teubner, Stuttgart, 1998.
29. J. Weickert, S. Ishikawa, and A. Imiya. Linear scale-space has first been proposed in Japan. *Journal of Mathematical Imaging and Vision*, 10(3):237–252, May 1999.
30. A. L. Yuille and N. M. Grzywacz. A computational theory for the perception of coherent visual motion. *Nature*, 333:71–74, 1988.
31. A. L. Yuille and N. M. Grzywacz. The motion coherence theory. In *Proc. Second International Conference on Computer Vision*, pages 344–354, Washington, DC, Dec. 1988. IEEE Computer Society Press.